Dynamics of a Spinning Space Station with a Counterweight Connected by Multiple Cables

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A derivation of the linear equations of motion of a conservative system consisting of two rigid bodies connected by an arbitrary number of massless cables with linear axial stiffness is presented. First, the nonlinear equations are derived using Newtonian mechanics. Then the equations of motion for each body are linearized while treating the cable effects as external forces. The nonlinear expressions for the cable forces and torques are expanded in a Taylor series about equilibrium values of the system coordinates. This expansion results in a cable stiffness matrix when second-order terms are neglected. A method of obtaining the equilibrium values of the coordinates is discussed, and results are presented. The linear model is verified, and some indication of the range of validity is determined by comparing time responses with a digital simulation of the nonlinear system.

Introduction

THE desirability, if not the necessity, to have artificial gravity in space for long-term missions has led to the study of spinning space stations or the utilization of centrifugal force as gravity. However, the Coriolis effect is an inherent problem with this concept. To minimize this effect, a low-spin rate in conjunction with a long distance from the center of rotation to the crew compartment is needed. The concept of using a cable-connected counterweight is attractive as a means of obtaining this long distance with a minimum weight penalty. Another attractive feature is that it would be possible to deploy the counterweight from the space station while in orbit by initiating the sping and "paying out" the cable.

For single cable systems, a significant amount of analytical work has been done in which attention is focused primarily on the dynamics of the cable.[1–3] The rather comprehensive work by Tai[4] includes a modal analysis of a specific multiple-cable configuration. The cables are assumed to be massless and have linear axial stiffness. By using multiple cables, as illustrated in Fig. 1, it is possible to achieve a more stable system. This is especially true with regard to motion about the axis along the direction of the cable. Formulating the equations of motion of such a system is rather straightforward if the mass of the cables is neglected and the space station and counterweight are assumed to be rigid bodies. There are several digital computer simulations which will handle this problem with arbitrary body mass characteristics, cable connection points, etc. However, digital simulation of the nonlinear equations is a very cumbersome way to analyze the system, because parameter studies must be made by making many computer runs. Therefore, it is desirable to have a linear description of the system to which other methods of analysis can be applied.

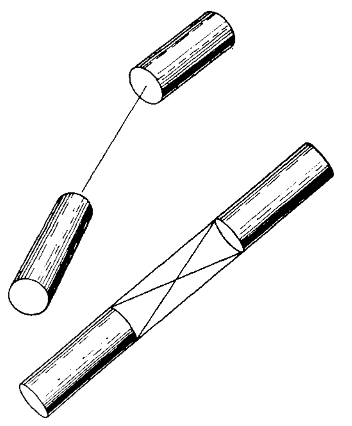

Fig. 1 Cable connected space stations.

This paper presents a derivation of the linear equations of motion by separating the system into several pieces, linearizing the pieces, and putting the system together again. By using this approach, the terms in the resulting equations can be directly identified with the physical parameters of the system. The laborious task of expanding matrix and vector products to identify nonlinear terms is minimized.

System Definition

The system to be considered consists of two rigid bodies connected by an arbitrary number of massless cables with linear axial stiffness and no damping. The mass characteristics of the bodies, the cable connection points, and individual cable stiffnesses are arbitrary. Orbital dynamics and external forces are excluded. No external torques were included; however, the resulting equations are not completely restricted in this sense. The external torque must not be such that it drives the system far from equilibrium. The elimination of external forces reduces the system from 12 to 9 degrees of freedom, since the coordinates of the system center of mass have no acceleration and need not be considered.

Figure 2 illustrates the coordinate systems and sign conventions chosen. (x_0, y_0, z_0) is an inertial reference frame with its origin at the system center of mass. (x_s, y_s, z_s) coincides with (x_0, y_0, z_0), except that it is defined to spin about its z-axis with a constant rate Ω. (x_1, y_1, z_1) and (x_2, y_2, z_2) are reference frames fixed in bodies 1 and 2 with origins at the respective centers of mass. The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ locate the centers of mass of the bodies with respect to the system center of mass. $\mathbf{a}_i$ and $\mathbf{b}_i$ locate the attachment points of the ith cable on each body. $\mathbf{l}_i$ is the vector length of the ith cable and is chosen to be positive when directed from body 1 to body 2. $\mathbf{F}_{1i}$ and $\mathbf{F}_{2i}$ are the forces of the respective bodies resulting from the ith cable. $\mathbf{R}$ locates the center of mass of body 2 with respect to body 1. I_1, m_1, I_2 and m_2 are the moment of inertia dyadics and masses of the bodies.

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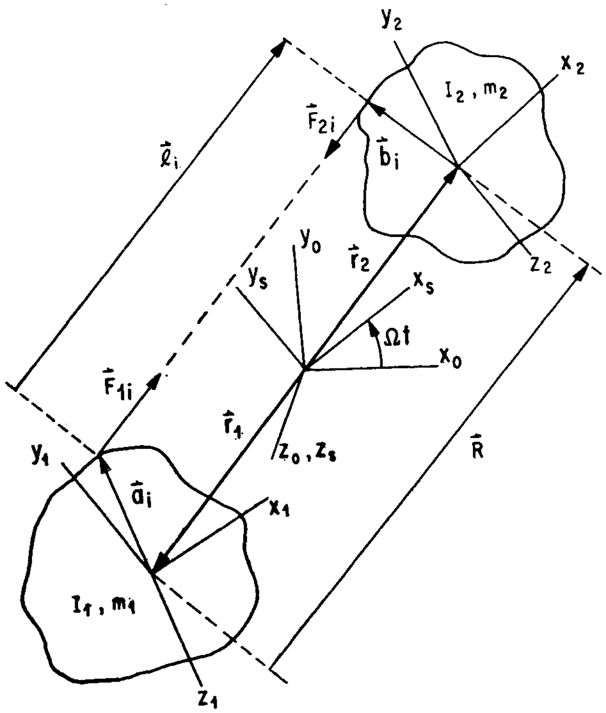

Fig. 2 System configuration.

Nonlinear Equations of Motion

Considering the ith cable to produce external torques $\mathbf{T}_{1i}$ and $\mathbf{T}_{2i}$ on the respective bodies, the rotational equations of motion are[5]

$$I_1 \cdot \dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times I_1 \cdot \boldsymbol{\omega}_1 = \sum_{i=1}^{N} \mathbf{T}_{1i} \tag{1}$$

and

$$I_2 \cdot \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times I_2 \cdot \boldsymbol{\omega}_2 = \sum_{i=1}^{N} \mathbf{T}_{2i} \tag{2}$$

where $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are the angular velocities of the body reference frames, and N is the number of cables. The translational equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = \sum_{i=1}^{N} \mathbf{F}_{1i} \tag{3}$$

and

$$m_2\ddot{\mathbf{r}}_2 = \sum_{i=1}^{N} \mathbf{F}_{2i} \tag{4}$$

The following relationships are obvious from Fig. 2

$$\mathbf{F}_{1i} = -\mathbf{F}_{2i} = \mathbf{F}_i \tag{5}$$

and

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1 \tag{6}$$

After combining Eqs. (3) and (4) and making use of relationships (5) and (6), the translational equation of motion becomes

$$\frac{m_1 m_2}{m_1 + m_2}\ddot{\mathbf{R}} + \sum_{i=1}^{N} \mathbf{F}_i = 0 \tag{7}$$

If $\ddot{\mathbf{R}}$ is to be evaluated, a frame of reference must be chosen for $\mathbf{R}$. If the spin frame is chosen, the angular velocity involved is constant by definition, and the choice to express $\mathbf{F}$ in the spin frame is automatically made. In matrix notation, Eq. (7) becomes

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} + \begin{bmatrix} 0 & -2m\Omega & 0 \\ 2m\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} + \begin{bmatrix} -m\Omega^2 & 0 & 0 \\ 0 & -m\Omega^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = 0 \tag{8}$$

where

$$m = m_1 m_2/(m_1 + m_2)$$

and

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i$$

(x,y,z) and (F_x, F_y, F_z) are the components of $\mathbf{R}$ and $\mathbf{F}$ resolved in the spin frame.

The orientations of the body 1 and body 2 frames with respect to the spin frame are defined by Euler angles (ϕ_1, ϕ_2, ϕ_3) and $(\theta_1, \theta_2, \theta_3)$, which result in the transformations $\mathbf{A}$ and $\mathbf{B}$ so that

$$\{V\}_1 = [\mathbf{A}]\{V\}_s \tag{9}$$

and

$$\{V\}_2 = [\mathbf{B}]\{V\}_s \tag{10}$$

where $\mathbf{V}$ is any vector. A (1,2,3) rotation sequence is used to generate the transformations. Since $\mathbf{F}_i$ is expressed in the spin frame, and $\mathbf{T}_{1i}$ and $\mathbf{T}_{2i}$ must be expressed in the body frames, we have

$$\mathbf{T}_{1i} = \mathbf{a}_i \times [\mathbf{A}]\{F_i\} \tag{11}$$

and

$$\mathbf{T}_{2i} = -\mathbf{b}_i \times [\mathbf{B}]\{F_i\} \tag{12}$$

where the components of $\mathbf{a}_i$ and $\mathbf{b}_i$ are expressed in the associated body frames.

Referring to Fig. 2, the cable forces are represented by

$$\mathbf{F}_i = k_i(l_i - l_{oi})(\mathbf{l}_i/l_i) \tag{13}$$

where $k_i = A_i E_i / l_{oi}$ and $l_i = |\mathbf{l}_i|$. A_i is the area of the cable cross section, E_i is Young's modulus of elasticity, and l_{oi} is the unstretched length of the ith cable. From the figure, the following vector relationship is obvious:

$$\mathbf{l}_i = \mathbf{R} + \mathbf{b}_i - \mathbf{a}_i \tag{14}$$

Since $\mathbf{l}_i$ must be expressed in the spin reference frame,

$$\{l_i\} = \{R\} + [\mathbf{B}^T]\{b_i\} - [\mathbf{A}^T]\{a_i\} \tag{15}$$

Equations (1, 2, 8, 11–13, and 15) essentially define the nonlinear equations of motion, and the derivation as such will not be pursued further.

Derivation of the Equilibrium Conditions

The angular velocities of the bodies are expressed as

$$\boldsymbol{\omega}_1 = \boldsymbol{\Omega}_1 + \boldsymbol{\omega}_1' \tag{16}$$

and

$$\boldsymbol{\omega}_2 = \boldsymbol{\Omega}_2 + \boldsymbol{\omega}_2' \tag{17}$$

where

$$\boldsymbol{\Omega}_1 = [\mathbf{A}]\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} \tag{18}$$

and

$$\boldsymbol{\Omega}_2 = [\mathbf{B}]\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} \tag{19}$$

and the primed angular velocities represent small deviations from the associated constant equilibrium angular velocity. If the system is at equilibrium, the transformations **A** and **B** are constant, and $\boldsymbol{\omega}_1'$ and $\boldsymbol{\omega}_2'$ are zero, or

$$\boldsymbol{\omega}_1 = \boldsymbol{\Omega}_1 \tag{20}$$

$$\boldsymbol{\omega}_2 = \boldsymbol{\Omega}_2 \tag{21}$$

and

$$\dot{\boldsymbol{\omega}}_1 = \dot{\boldsymbol{\omega}}_2 = 0 \tag{22}$$

Substituting these conditions into Eqs. (1) and (2) yields the following conditions for rotational equilibrium

$$\boldsymbol{\Omega}_1 \times I_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{T}_1 = 0 \tag{23}$$

and

$$\boldsymbol{\Omega}_2 \times I_2 \cdot \boldsymbol{\Omega}_2 - \mathbf{T}_2 = 0 \tag{24}$$

where

$$\mathbf{T}_1 = \sum_{i=1}^{N} \mathbf{T}_{1i}$$

and

$$\mathbf{T}_2 = \sum_{i=1}^{N} \mathbf{T}_{2i}$$

The condition for translational equilibrium from Eq. (8) is

$$\begin{Bmatrix} F_x - m\Omega^2 x \\ F_y - m\Omega^2 y \\ F_z \end{Bmatrix} = 0 \tag{25}$$

The simultaneous solution of Eqs. (23–25) yields the equilibrium values of $\{\phi\}$, $\{R\}$, and $\{\theta\}$.

Solution of the Equilibrium Conditions

The problem to be solved may be stated as

$$\begin{Bmatrix} \boldsymbol{\Omega}_1 \times I_1 \cdot \boldsymbol{\Omega}_1 - \mathbf{T}_1 \\ \hline F_x - m\Omega^2 x \\ F_y - m\Omega^2 y \\ F_z \\ \hline \boldsymbol{\Omega}_2 \times I_2 \cdot \boldsymbol{\Omega}_2 - \mathbf{T}_2 \end{Bmatrix} = 0 \tag{26}$$

The solution to this equation is obvious for the special case where body axes are principal axes of inertia and the cables have equal stiffnesses and are symmetrically attached. Under these conditions the equilibrium angles are zero, and the problem is reduced to one of calculating the cable stretch required to make the tension **F** equal to the centripetal acceleration. The general solution, however, is somewhat more complicated. Physically, the equilibrium positions will be such that the complete system spins about a principal axis of inertia, and the centripetal acceleration is balanced by the resultant cable tension. The individual bodies will not, in general, spin about principal axes; but the dynamic torque is constant in the body frame, and is balanced by the resultant cable torque.

It is obvious that the problem as described by Eq. (26) involves more variables than are necessary to describe the equilibrium. The z-coordinate must always be zero at equilibrium because the centers of mass must lie in the spin plane. From Fig. 2 it can be seen that a rotation of the complete system through a constant angle in the spin plane has no physical effect on the equilibrium. There are no other obvious simplifications to the problem.

Equation (26) is solved numerically with the use of a digital computer by the Newton-Raphson technique, which involves an $n \times n$ matrix of partial derivatives for n variables. Since all nine variables are needed in the final equation of motion, the partial derivatives needed here can be directly applied later if all the variables are used. Equation (26) can be restated as

$$\{\mathbf{f}(\mathbf{p})\} = 0 \tag{27}$$

where

$$\mathbf{p} = \begin{Bmatrix} \boldsymbol{\phi} \\ \hline \mathbf{R} \\ \hline \boldsymbol{\phi} \end{Bmatrix} \tag{28}$$

Equation (27) can be solved numerically using the recursive formula

$$\mathbf{p}_{i+1} = \mathbf{p}_i - [\nabla|_{\mathbf{p}_i}]^{-1}\{\mathbf{f}(\mathbf{p}_i)\} \tag{29}$$

where i indicates the ith iterated value. ∇ is a 9×9 matrix of partial derivatives of the elements of **f** with respect to the elements of **p**.

∇ is constructed by partitioning the 9×9 matrix into nine 3×3 submatrices which are treated independently for derivation of the expressions for the partial derivatives. Each submatrix is further broken down into the sum of partial derivatives of dynamic and cable contributions to **f**, or

$$\nabla = \nabla_c + \nabla_d \tag{30}$$

The contribution of the cables to ∇ is represented by the expression

$$\nabla_c = \left[\begin{array}{c|c|c} -(\partial \mathbf{T}_1/\partial \boldsymbol{\phi}) & -(\partial \mathbf{T}_1/\partial \mathbf{R}) & -(\partial \mathbf{T}_1/\partial \boldsymbol{\theta}) \\ \hline \partial \mathbf{F}/\partial \boldsymbol{\phi} & \partial \mathbf{F}/\partial \mathbf{R} & \partial \mathbf{F}/\partial \boldsymbol{\theta} \\ \hline -(\partial \mathbf{T}_2/\partial \boldsymbol{\phi}) & -(\partial \mathbf{T}_2/\partial \mathbf{R}) & -(\partial \mathbf{T}_2/\partial \boldsymbol{\theta}) \end{array}\right] \tag{31}$$

The dynamic contribution to ∇ is given by

$$\nabla_d = \left[\begin{array}{c|ccc|c} \partial \mathbf{T}_{d1}/\partial \boldsymbol{\phi} & & 0 & & 0 \\ \hline & -m\Omega^2 & 0 & 0 & \\ 0 & 0 & -m\Omega^2 & 0 & 0 \\ & 0 & 0 & 0 & \\ \hline 0 & & 0 & & \partial \mathbf{T}_{d2}/\partial \boldsymbol{\theta} \end{array}\right] \tag{32}$$

where

$$\mathbf{T}_{d1} = \boldsymbol{\Omega}_1 \times I_1 \cdot \boldsymbol{\Omega}_1 \tag{33}$$

and

$$\mathbf{T}_{d2} = \boldsymbol{\Omega}_2 \times I_2 \cdot \boldsymbol{\Omega}_2 \tag{34}$$

Considering Eqs. (33) and (34), some manipulations will be performed to facilitate taking the partial derivatives. Changing to indicial notation and changing all present subscripts to superscripts, Eq. (33) becomes

$$T_i^{(d1)} = \varepsilon_{ijk}\Omega_j^{(1)} I_{kl}^{(1)} \Omega_l^{(1)} \tag{35}$$

where

$$\varepsilon_{ijk} = \begin{cases} 1, \text{ cyclic } i,j,k \\ 0, \text{ repeated } i,j,k \\ -1, \text{ anticyclic } i,j,k \end{cases} \tag{36}$$

and all subscripts vary from 1 to 3. Applying the same procedure to Eq. (18), we have

$$\Omega_i^{(1)} = A_{ij}\Omega_j = \Omega A_{i3} \tag{37}$$

Combining Eqs. (35) and (37), the ith component of $\mathbf{T}_{d1}$ is given by

$$T_i^{(d1)} = \Omega^2 \varepsilon_{ijk} A_{j3} I_{kl}^{(1)} A_{l3} \tag{38}$$

Similarly,

$$T_i^{(d2)} = \Omega^2 \varepsilon_{ijk} B_{j3} I_{kl}^{(2)} B_{l3} \tag{39}$$

Taking the appropriate partial derivatives,

$$\frac{\partial T_i^{(d1)}}{\partial \phi_m} = \Omega^2 \varepsilon_{ijk} \left[\frac{\partial A_{j3}}{\partial \phi_m} I_{kl}^{(1)} {}_l A_{l3} + A_{j3} I_{kl}^{(1)} \frac{\partial A_{l3}}{\partial \phi_m} \right] \tag{40}$$

or

$$\frac{\partial T_i^{(d1)}}{\partial \phi_m} = \Omega^2 \varepsilon_{ijk} I_{kl}^{(1)} (P_{j3m} A_{l3} + A_{j3} P_{l3m}) \tag{41}$$

where

$$P_{ijk} = \partial A_{ij} / \partial \phi_k \tag{42}$$

Similarly,

$$\frac{\partial T_i^{(d2)}}{\partial \theta_m} = \Omega^2 \varepsilon_{ijk} I_{kl}^{(2)} (Q_{j3m} B_{l3} + B_{j3} Q_{l3m}) \tag{43}$$

where

$$Q_{ijk} = \partial B_{ij} / \partial \theta_k \tag{44}$$

Equations (41) and (43) represent the (1,1) and (3,3) submatrices of ∇_d, and are in a form which can be readily programed on a digital computer. P_{ijk} and Q_{ijk} have particular expressions associated with each set of subscript values. These expressions are obtained directly by application of the definitions of P_{ijk} and Q_{ijk}, and can be programed to be evaluated as necessary. Since the (2,2) submatrix of ∇_d is given in final form in Eq. (32) and all of the other elements are zero, the derivation of ∇_d is complete.

Dropping the subscript indicating a particular cable and changing to indicial notation, Eq. (13) becomes

$$F_i = k[1 - (l_o/l)] l_i \tag{45}$$

The (2,2) submatrix of ∇_c is represented by

$$\frac{\partial F_i}{\partial R_j} = k \left[\left(1 - \frac{l_o}{l}\right) \frac{\partial l_i}{\partial R_j} + \frac{\partial}{\partial R_j} \left(1 - \frac{l_o}{l}\right) l_i \right] \tag{46}$$

or

$$\frac{\partial F_i}{\partial R_j} = k \left[\left(1 - \frac{l_o}{l}\right) \frac{\partial l_i}{\partial R_j} + \frac{l_o l_i}{l^2} \left(\frac{\partial l}{\partial R_j} \right) \right] \tag{47}$$

Changing Eq. (15) to indicial notation, we have

$$l_i = R_i + B_{ji} b_j - A_{ji} a_j \tag{48}$$

Then

$$\partial l_i / \partial R_j = \delta_{ij} \tag{49}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{50}$$

The length of the cable may be written as

$$l = (l_i l_i)^{1/2} \tag{51}$$

Then

$$\frac{\partial l}{\partial R_j} = \frac{1}{2} (l_i l_i)^{-1/2} \left[l_i \frac{\partial l_i}{\partial R_j} + \frac{\partial l_i}{\partial R_j} l_i \right] \tag{52}$$

or

$$\frac{\partial l}{\partial R_j} = \frac{1}{l} \left(l_i \frac{\partial l_i}{\partial R_j} \right) = \frac{l_i}{l} \delta_{ij} = \frac{l_j}{l} \tag{53}$$

Substituting Eqs. (49) and (53) into Eq. (47) gives the final expression

$$\frac{\partial F_i}{\partial R_j} = k \left[\left(1 - \frac{l_o}{l}\right) \delta_{ij} + \frac{l_o l_i l_j}{l^3} \right] \tag{54}$$

This equation is of a usable form since the length of the cable is easily calculated from Eqs. (48) and (51).

In a similar manner

$$\frac{\partial F_l}{\partial \phi_m} = k \left[\left(1 - \frac{l_o}{l}\right) \frac{\partial l_l}{\partial \phi_m} + \frac{l_o l_l}{l^2} \frac{\partial l}{\partial \phi_m} \right] \tag{55}$$

$$\partial l_l / \partial \phi_m = -P_{jlm} a_j \tag{56}$$

and

$$\partial l / \partial \phi_m = (-l_i a_j / l) P_{jim} \tag{57}$$

Then

$$\frac{\partial F_l}{\partial \phi_m} = -k \left[\left(1 - \frac{l_o}{l}\right) P_{jlm} a_j + \frac{l_o l_l}{l^3} (l_i a_j P_{jim}) \right] \tag{58}$$

Similarly,

$$\frac{\partial F_l}{\partial \theta_m} = k \left[\left(1 - \frac{l_o}{l}\right) Q_{jlm} b_j + \frac{l_o l_l}{l^3} (l_i b_j Q_{jim}) \right] \tag{59}$$

Considering the cable torques, Eqs. (11) and (12) are written in indicial notation and become

$$T_i^{(1)} = \varepsilon_{ijk} a_j A_{kl} F_l \tag{60}$$

and

$$T_i^{(2)} = -\varepsilon_{ijk} b_j B_{kl} F_l \tag{61}$$

The remaining submatrices are found in terms of those already derived, and are listed as follows:

$$\partial T_i^{(1)} / \partial R_m = \varepsilon_{ijk} a_j A_{kl} (\partial F_l / \partial R_m) \tag{62}$$

$$\partial T_i^{(2)} / \partial R_m = -\varepsilon_{ijk} b_j B_{kl} (\partial F_l / \partial R_m) \tag{63}$$

$$\partial T_i^{(1)} / \partial \theta_m = \varepsilon_{ijk} a_j A_{kl} (\partial F_l / \partial \theta_m) \tag{64}$$

$$\partial T_i^{(2)} / \partial \phi_m = -\varepsilon_{ijk} b_j B_{kl} (\partial F_l / \partial \phi_m) \tag{65}$$

$$\frac{\partial T_i^{(1)}}{\partial \phi_m} = \varepsilon_{ijk} a_j \left[A_{kl} \frac{\partial F_l}{\partial \phi_m} + F_l P_{klm} \right] \tag{66}$$

and

$$\frac{\partial T_i^{(2)}}{\partial \theta_m} = -\varepsilon_{ijk} b_j \left[B_{kl} \frac{\partial F_l}{\partial \theta_m} + F_l Q_{klm} \right] \tag{67}$$

Equations (54, 58, 59, and 62–67) complete the derivation of ∇_c. It should be noted that these equations apply to one cable at a time, but for more than one the contributions of each are simply summed to give the resultant effect.

A digital computer program was written to solve Eq. (26) via the process outlined. The possibility of the existence of more than one physically distinct set of equilibrium coordinates has been explored, but the solution seems to be unique except for the influence of the previously described rotation of the complete system in the spin plane. If the direction of the cables is generally along the x-body axes, the equilibrium value of y usually comes out close to zero if the initial guess to start the iteration is not too far away. It is obvious from the program results that y is unimportant with regard to equilibrium. Correspondingly, the difference between the angles ϕ_3 and θ_3 is the important factor rather than the absolute value of either. For realistic cases where the products of inertia are relatively small, etc., convergence is rapidly achieved with the "unstretched" values of the coordinates as an initial guess.

Linearization of the Equations of Motion

The translational equation of motion, Eq. (8), is already linear if the cable tension **F** is linear. Therefore, the rotational equations will be examined. Since Eqs. (1) and (2) are of the same form, only Eq. (1) is considered, and the subscript 1 is temporarily dropped for convenience. The total angular velocity of body 1 may be written as

$$\{\omega\} = [\mathbf{A}]\begin{Bmatrix}0\\0\\\Omega\end{Bmatrix} + [\mathbf{S}]\{\dot{\phi}\} \tag{68}$$

where

$$[\mathbf{S}] = \begin{bmatrix}\cos\phi_3\cos\phi_2 & \sin\phi_3 & 0\\ -\sin\phi_3\cos\phi_2 & \cos\phi_3 & 0\\ \sin\phi_2 & 0 & 1\end{bmatrix} \tag{69}$$

which is the inverse of the more familiar transformation from body angular velocities to Euler angle rates for a (1,2,3) rotation sequence.

At this point, a choice must be made with regard to the exact definition of the small angles which describe the motion of the body about equilibrium. One possibility is that the orientation of the body be described by three Euler angles which are each made up of a constant equilibrium value plus a small deviation. In this case, the small angles are "embedded" in the transformation **A** and are not about orthogonal axes. Also, **S** involves the equilibrium values of the angles. Another possibility is that **A** be the product of two transformations, one of which is constant and the other is restricted to small angles. This can be thought of as defining an additional reference frame which is coincident with the body frame at equilibrium. Under these conditions, the small angles are essentially about body axes, and **S** involves the small angles only. The effects of the first possibility would be compensating, but it seems to be more reasonable to choose the second one. Then **A** is expressed by

$$[\mathbf{A}] = \begin{bmatrix}1 & \phi_3' & -\phi_2'\\ -\phi_3' & 1 & \phi_1'\\ \phi_2' & -\phi_1' & 1\end{bmatrix}[\mathbf{A}_e], \tag{70}$$

or

$$[\mathbf{A}] = [E - \tilde{\phi}'][\mathbf{A}_e], \tag{71}$$

where $\mathbf{A}_e$ is constant, $\mathbf{E}$ is the identity matrix, and $\tilde{\phi}'$ is the conventional skew-symmetric matrix associated with the vector $\boldsymbol{\phi}'$. If the expression for $\boldsymbol{\omega}$ is to be linear, **S** must be considered to be the identity matrix. Equation (68) can be written now as

$$\boldsymbol{\omega} = \boldsymbol{\Omega}_e - \boldsymbol{\phi}' \times \boldsymbol{\Omega}_e + \dot{\boldsymbol{\phi}} \tag{72}$$

where

$$\boldsymbol{\Omega}_e = [\mathbf{A}_e]\begin{Bmatrix}0\\0\\\Omega\end{Bmatrix} \tag{73}$$

Now

$$\boldsymbol{\omega} = \boldsymbol{\Omega}_e + \dot{\boldsymbol{\phi}}' + \boldsymbol{\Omega}_e \times \boldsymbol{\phi}' \tag{74}$$

and

$$\dot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\phi}}' + \boldsymbol{\Omega}_e \times \dot{\boldsymbol{\phi}}' \tag{75}$$

Substituting these expressions for $\boldsymbol{\omega}_1$ and $\dot{\boldsymbol{\omega}}_1$ in Eq. (1) and deleting nonlinear terms gives the linear rotational equation of motion in vector form

$$\begin{aligned} I\cdot\ddot{\boldsymbol{\phi}}' &+ I\cdot(\boldsymbol{\Omega}_e\times\dot{\boldsymbol{\phi}}')\\ &+ \dot{\boldsymbol{\phi}}'\times I\cdot\boldsymbol{\Omega}_e + \boldsymbol{\Omega}_e\times I\cdot\dot{\boldsymbol{\phi}}' + \boldsymbol{\Omega}_e\times I\cdot\boldsymbol{\Omega}_e\\ &+ \boldsymbol{\Omega}_e\times I\cdot(\boldsymbol{\Omega}_e\times\boldsymbol{\phi}') + (\boldsymbol{\Omega}_e\times\boldsymbol{\phi}')\times I\cdot\boldsymbol{\Omega}_e = \mathbf{T} \end{aligned} \tag{76}$$

In matrix notation, after some rearranging of terms, this can be written as

$$\begin{aligned}[I]\{\ddot{\phi}'\} &+ [I\tilde{\Omega}_e + \tilde{\Omega}_e I - \tilde{H}_e]\{\dot{\phi}'\}\\ &+ [\tilde{\Omega}_e I\tilde{\Omega}_e - \tilde{H}_e\tilde{\Omega}_e]\{\phi'\} + \{\tilde{\Omega}_e I\tilde{\Omega}_e - T\} = 0\end{aligned} \tag{77}$$

where

$$H_e = I\cdot\boldsymbol{\Omega}_e \tag{78}$$

The complete system can now be pulled together as

$$[\mathbf{M}]\{\ddot{q}\} + [\mathbf{C}]\{\dot{q}\} + [\mathbf{K}_d]\{\mathbf{q}\} + \{\mathbf{f}\} = 0 \tag{79}$$

where

$$\mathbf{q} = \begin{Bmatrix}\boldsymbol{\phi}'\\ \hdashline \mathbf{r}\\ \hdashline \boldsymbol{\theta}'\end{Bmatrix} \tag{80}$$

and $\mathbf{r} = \mathbf{R} - \mathbf{R}_e$. $\mathbf{R}_e$ is the equilibrium value of $\mathbf{R}$. The coefficient matrices are defined as follows:

$$\mathbf{M} = \left[\begin{array}{c:ccc:c} I_1 & & 0 & & 0\\ \hdashline & m & 0 & 0 & \\ 0 & 0 & m & 0 & 0\\ & 0 & 0 & m & \\ \hdashline 0 & & 0 & & I_2\end{array}\right] \tag{81}$$

$\mathbf{C} =$

$$\left[\begin{array}{c:ccc:c} I_1\tilde{\Omega}_{1e}+\tilde{\Omega}_{1e}I_1-\tilde{H}_{1e} & & & & 0\\ \hdashline & 0 & -2m\Omega & 0 & \\ 0 & 2m\Omega & 0 & 0 & 0\\ & 0 & 0 & 0 & \\ \hdashline 0 & & 0 & & I_2\tilde{\Omega}_{2e}+\tilde{\Omega}_{2e}I_2-\tilde{H}_{2e}\end{array}\right] \tag{82}$$

$\mathbf{K}_d =$

$$\left[\begin{array}{c:ccc:c} \tilde{\Omega}_{1e}I_1\tilde{\Omega}_{1e}-\tilde{H}_{1e}\tilde{\Omega}_{1e} & & 0 & & 0\\ \hdashline & -m\Omega^2 & 0 & 0 & \\ 0 & 0 & -m\Omega^2 & 0 & 0\\ & 0 & 0 & 0 & \\ \hdashline 0 & & 0 & & \tilde{\Omega}_{2e}I_2\tilde{\Omega}_{2e}-\tilde{H}_{2e}\tilde{\Omega}_{2e}\end{array}\right] \tag{83}$$

Also

$$\mathbf{f} = \begin{Bmatrix}\tilde{\Omega}_{1e}I_1\Omega_{1e} - T_1\\ \hdashline F_x - m\Omega^2 x_e\\ F_y - m\Omega^2 y_e\\ F_z\\ \hdashline \tilde{\Omega}_{2e}I_2\Omega_{2e} - T_2\end{Bmatrix} \tag{84}$$

which is immediately recognized as the function from which the equilibrium positions were determined, and is known to be zero when evaluated at equilibrium. The dynamic quantities in **f** are not functions of **q**, the generalized displacement from equilibrium. However, the cable forces and torques are functions of **q**, and must be dealt with accordingly. If the cable forces and torques are expanded in a Taylor series about equilibrium (zero) values of **q**, a cable stiffness matrix results when second-order terms are neglected. ∇_c in Eq. (31) is essentially the stiffness matrix needed when evaluated at equilibrium, which is automatically done when solving for the equilibrium coordinates **p**. However, **p** involves $\boldsymbol{\phi}$, **R**, and $\boldsymbol{\theta}$, while **q** involves ϕ', **r**, and $\boldsymbol{\theta}'$. The angles in **q** are about body axes, but those in **p** are not. Therefore, an alteration must be made to ∇_c before it can be used. ∇_c in its present form would apply to small deviations of the Euler angles about their equilibrium values. These small deviations are related to small deviations $\boldsymbol{\phi}'$ about body axes in the same way that

Euler angle rates are related to body angular velocities, which is defined by **S** in Eq. (69). The relationships between the angles are given by

$$\{\phi\} = [\mathbf{S}^{-1}]\{\phi'\} \tag{85}$$

and

$$\{\theta\} = [\mathbf{U}^{-1}]\{\theta'\} \tag{86}$$

where **U** corresponds to **S** but involves $\boldsymbol{\theta}$ rather than $\boldsymbol{\phi}$. If **S** and **U** are evaluated at equilibrium, the desired relationships are established. These relationships are incorporated in ∇_c by post multiplying each 3×3 submatrix involving $\boldsymbol{\phi}$ by $\mathbf{S}^{-1}$, and those involving $\boldsymbol{\theta}$ by $\mathbf{U}^{-1}$. If the altered ∇_c is represented by $\mathbf{K}_c$, and $\mathbf{K} = \mathbf{K}_d + \mathbf{K}_c$, Eq. (79) can be put in the final form

$$[\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{C}]\{\dot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} = 0 \tag{87}$$

which represents the linear model of the nine-degree-of-freedom system.

Comparison with the Nonlinear Model

The nominal spin rate is affected somewhat by the initial conditions. In the nonlinear model, this means that the spin rate simply increases or decreases slightly from the intended value. In the linear model, this means that the generalized velocities combine in such a way that a "rigid body" rotation of constant rate relative to the spin frame occurs. However, the system is still in a state of dynamic equilibrium, but at a spin rate other than Ω. The constant drift relative to the spin frame eventually causes trouble because the angles do not remain small. The initial conditions which cause the drift to occur are those which give the system more or less angular momentum about the z-axis than it would have at equilibrium with a spin rate of Ω. However, this problem can be avoided by choosing initial conditions which do not produce a resultant component of angular momentum in the z-direction. Since the nominal spin rate is arbitrary, this does not degrade the generality of the linear model.

Figure 3 illustrates the configuration chosen for a direct numerical comparison between the linear and nonlinear models. There are four cables, and they are connected as shown by the points 1, 2, 3 and 4. All of the cables have the same unstretched length, area of cross section, and modulus of elasticity. The dimensions are shown in the figure, and the remaining data is given below

$$I_1 = I_2 = \begin{bmatrix} 4 \times 10^5 & 0 & 0 \\ 0 & 1.7 \times 10^6 & 0 \\ 0 & 0 & 1.9 \times 10^6 \end{bmatrix} \text{ slug ft}^2$$

$$m_1 = m_2 = 140{,}000 \text{ lb}$$

$$AE = 50{,}000 \text{ lb}$$

$$\Omega = 6 \text{ deg/sec}$$

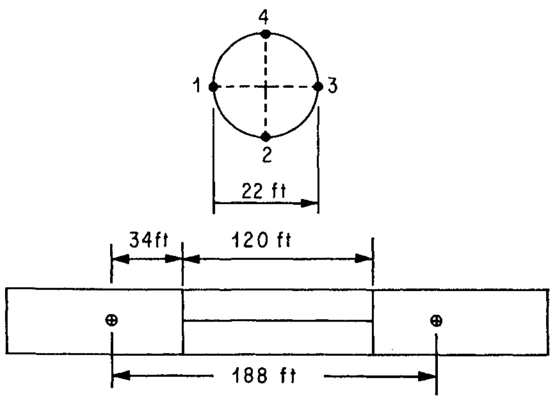

Fig. 3 Configuration for comparison.

For purposes of comparison, some changes had to be made on the output of the computer program generating time responses of the linear system from Eq. (87). The z-axis angles are measured relative to a line drawn through the centers of mass, and the translational coordinates are combined so that only the distance between the centers of mass is shown.

Figure 4 shows a comparison between time responses for this case. The initial conditions are 0.1 deg/sec about each axis on body 1, and -0.1 deg/sec about each axis on body 2. R_e, the equilibrium value of R, is 190.68 ft, and the initial condition is 190.95 ft. All other initial conditions are zero. As seen from the figure, the time responses for this case compare quite well. The small differences in frequency etc., are considered negligible, since a point-for-point comparison would not be expected, even if digital roundoff and numerical integration errors were not involved.

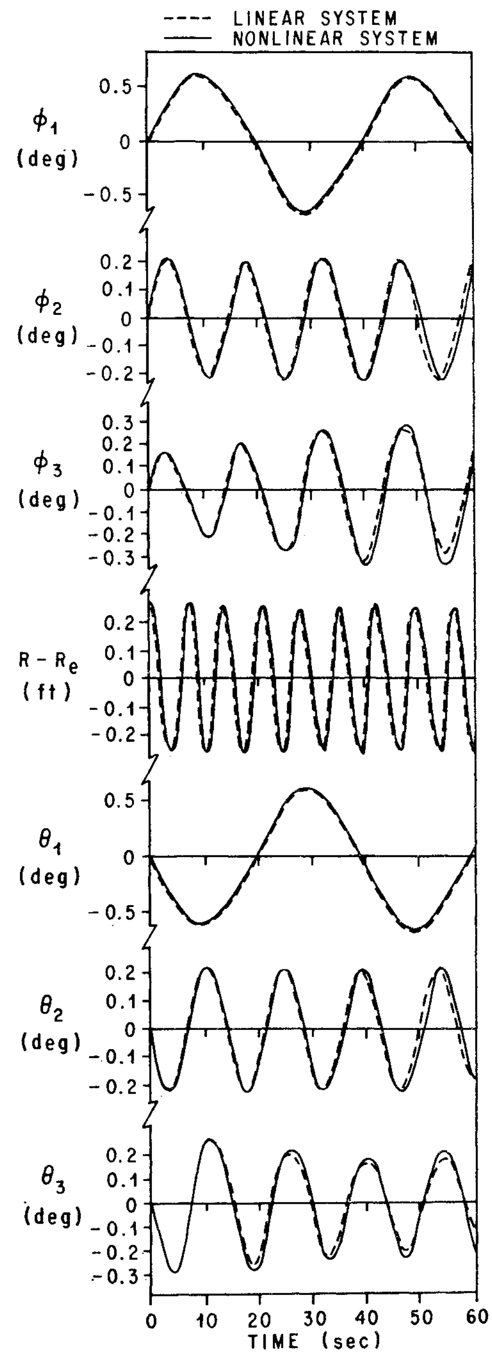

Fig. 4 Case I time responses.

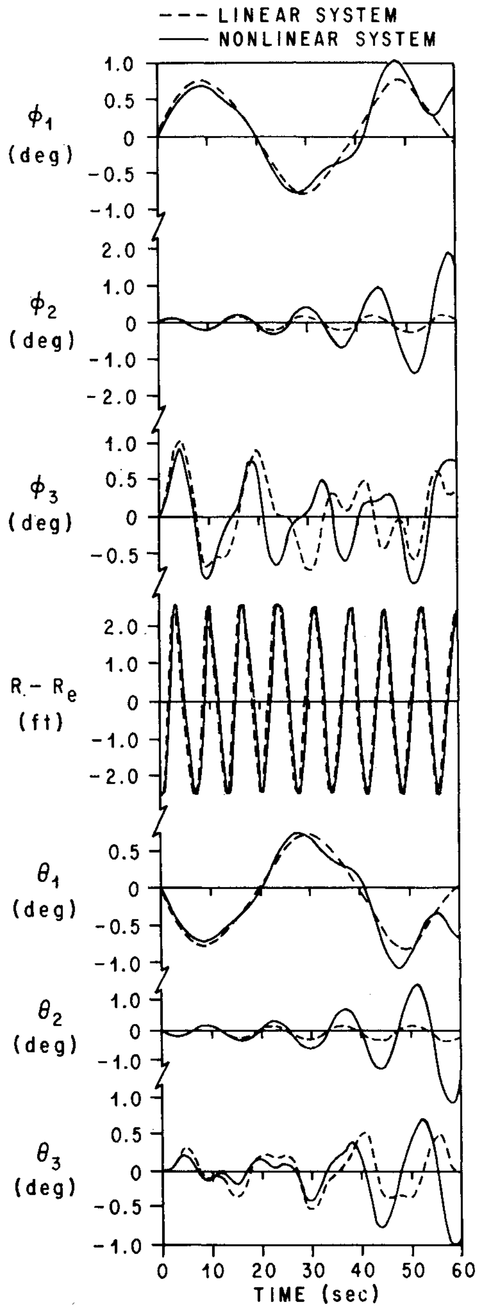

Fig. 5 Case II time responses.

Figure 5 shows a comparison for the same case except the initial value of R is 188 ft, or the value when the cables are unstretched. As seen from the figure, the comparison is poor. The prevalent nonlinear effect is the coupling of the time-varying part of the cable tension with the x-and y-axis body angles to produce a torque on the body. In this respect, the linear model can consider only the equilibrium value of the cable tension. In the linear sense, the time-varying part of the cable tension can produce a torque only if its direction is other than through the center of mass at equilibrium.

Conclusions

The linear model is valid for small deviations about equilibrium. "Small" in this case probably means that 1° or 2° angular excursions would be acceptable. Exactly what "small" means with regard to the translational motion is not as easy to determine, but 10% of the equilibrium cable stretch seems to be a safe estimate. The range of validity is actually of minor significance, since the linear model probably would not be used to produce time responses, but would be used as an aid in analyzing the nonlinear system. The ability to determine dynamic equilibrium states for the general case is very useful in analysis of the nonlinear system.

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